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Off-shell indefinite-metric triple-bracket generalization of the Dirac equation *

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Abstract

We present an off-shell indefinite-metric reformulation of the earlier on-shell positive-metric triple bracket generalization of the Dirac equation [1, 2]. The new version of the formalism solves the question of its manifest covariance.

1 Hamilton, Lie-Poisson and Lie-Nambu versions of the off-shell Dirac equation

In linear and pure-state case the standard positive metric associated with the Dirac equation is constructed by means of a spacelike hyperplane Σ but the continuity equation guarantees that the metric is in fact Σ-independent. In the generalized density-matrix nonlinear formulation we cannot use this argument and hence the independence of the whole formalism from the choice of Σ is unclear. The natural way out of the difficulty is to simply use the indefinite metric formulation which does not depend on any hyperplane. We therefore obtain a formalism which is manifestly covariant. The convention we use assumes that repeated Greek indices imply simultaneous summation over bispinor and integration over spacetime indices.

We begin with the off-shell version of the spinor form of the (free) Dirac equation

$$\sqrt{2}\nabla_{BA'}\psi^B = \partial_s\psi_{A'}; \quad \sqrt{2}\nabla_{AB'}\psi^{B'} = -\partial_s\psi_A. \tag{1}$$

Here ∂_s denotes a partial derivative with respect to a "proper time" which is conjugate to mass [3]. The Hamiltonian function ("average mass") is given by

$$H = \sqrt{2}\int d^4x\Big(\bar\psi^{A'} i\nabla_{BA'}\psi^B + \bar\psi^{A} i\nabla_{AB'}\psi^{B'}\Big) = \int d^4x\bar\psi^{\alpha'}(x)g_{\alpha'}{}^{\beta}i\nabla_\beta{}^\gamma\psi_\gamma(x) = \bar H \tag{2}$$

and leads to Hamilton equations equivalent to the Dirac equation:

$$i\partial_s\psi_\alpha = -g_{\alpha\alpha'}\frac{\delta H}{\delta\bar\psi_{\alpha'}}; \quad i\partial_s\bar\psi_{\alpha'} = g_{\alpha\alpha'}\frac{\delta H}{\delta\psi_\alpha}. \tag{3}$$

The abstract index bispinor convention is explained in the Appendix. The Poisson tensor and the symplectic form are given by $I_a = -g_{\alpha\alpha'}$ and $\omega^a = -g^{\alpha\alpha'}$ respectively. With these identifications and following step by step the scheme discussed in [1, 2] we obtain the Lie-Poisson and Lie-Nambu structures in their off-shell and indefinite-metric form.

1.1 Metric and higher order metric tensors

Metric tensors allowing to raise and lower indices in the infinite-dimensional Lie algebra are

$$g^{ab} = g^{\alpha\beta'}g^{\beta\alpha'}\delta(a-b')\delta(a'-b); \quad g_{ab} = g_{\alpha\beta'}g_{\beta\alpha'}\delta(a-b')\delta(a'-b). \tag{4}$$

The two tensors are symmetric and satisfy $g^{ab}g_{bc} = \varepsilon_\gamma{}^\alpha\varepsilon_{\gamma'}{}^{\alpha'} = \varepsilon_c{}^a$. Skipping the Dirac deltas we define higher order metric tensors which will be used in Casimir invariants:

$$g^{a_1\dots a_n} = g^{\alpha_1\alpha'_n}g^{\alpha_2\alpha'_1}g^{\alpha_3\alpha'_2}\dots g^{\alpha_{n-1}\alpha'_{n-2}}g^{\alpha_n\alpha'_{n-1}}, \tag{5}$$

$$g_{a_1\dots a_n} = g_{\alpha_1\alpha'_n}g_{\alpha_2\alpha'_1}g_{\alpha_3\alpha'_2}\cdots g_{\alpha_{n-1}\alpha'_{n-2}}g_{\alpha_n\alpha'_{n-1}}. \tag{6}$$

The case $n=1$ corresponds to the Poisson tensor and its inverse.

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1.2 Poisson and Lie-Poisson brackets

The Hamilton equations imply the Poisson bracket equations

$$\begin{aligned} i\,\partial_s F &= -g_{\alpha\alpha'}\Big(\frac{\delta F}{\delta\psi_\alpha}\frac{\delta H}{\delta\bar\psi_{\alpha'}} - \frac{\delta H}{\delta\psi_\alpha}\frac{\delta F}{\delta\bar\psi_{\alpha'}}\Big) && (7)\\ &= -g_{\alpha\beta'}\rho_{\beta\alpha'}\Big(\frac{\delta F}{\delta\rho_{\alpha\alpha'}}\frac{\delta H}{\delta\rho_{\beta\beta'}} - \frac{\delta H}{\delta\rho_{\alpha\alpha'}}\frac{\delta F}{\delta\rho_{\beta\beta'}}\Big) = \rho_a\Omega^a{}_{bc}\frac{\delta F}{\delta\rho_b}\frac{\delta H}{\delta\rho_c}. && (8)\end{aligned}$$

The structure constants are

$$\begin{aligned} \Omega^a{}_{bc} &= \varepsilon_{\gamma'}{}^{\alpha'}\varepsilon_\beta{}^\alpha g_{\gamma\beta'} - \varepsilon_{\beta'}{}^{\alpha'}\varepsilon_\gamma{}^\alpha g_{\beta\gamma'} && (9)\\ \Omega_{abc} &= g_{ad}\Omega^d{}_{bc} = -g_{\alpha\beta'}g_{\beta\gamma'}g_{\gamma\alpha'} + g_{\alpha\gamma'}g_{\beta\alpha'}g_{\gamma\beta'} && (10)\\ \Omega^{abc} &= g^{bd}g^{ce}\Omega^a{}_{de} = g^{\alpha\beta'}g^{\beta\gamma'}g^{\gamma\alpha'} - g^{\alpha\gamma'}g^{\beta\alpha'}g^{\gamma\beta'}. && (11)\end{aligned}$$

1.3 Lie-Nambu bracket form of linear proper time dynamics

Denote $S = S[\rho] = S(C_2[\rho]) = g^{ab}\rho_a\rho_b/2 =: C_2[\rho]/2$. The triple Lie-Nambu bracket form of dynamics is

$$i\partial_s F = \{F, H, S\} = \Omega_{abc}\frac{\delta F}{\delta\rho_a}\frac{\delta H}{\delta\rho_b}\frac{\delta S}{\delta\rho_c}. \qquad (12)$$

For $F = \rho_d$ Eq. (12) is the linear Liouville-von Neumann equation in its proper time version provided S is second-order in ρ_a.

2 Nonlinear generalization

2.1 Casimir invariants

Proofs of the theorems given below are analogous to those from [2] so we do not present them. Denote $g^{a_1\dots a_n}\rho_{a_1}\dots\rho_{a_n} =: C_n[\rho]$.

Theorem 1

$$\{C_n, C_m, \cdot\} = 0. \qquad (13)$$

C_n are therefore Casimir invariants for all Lie-Nambu brackets.

Theorem 2 *Let $S = S(C_1,\dots C_k,\dots)$ be any differentiable function of $C_1,\dots C_k\dots$, and H, F arbitrary (in general nonlinear) observables. Then*

$$\begin{aligned} \{C_n, F, S\} &= 0, && (14)\\ \partial_s C_n = -i\{C_n, H, S\} &= 0. && (15)\end{aligned}$$

2.2 N particles and separation of subsystems

Let $g^{N\,ab} = g^{a_1b_1}\dots g^{a_Nb_N}$, $g^N{}_{ab} = g_{a_1b_1}\dots g_{a_Nb_N}$. Consider an N-particle density matrix $\rho^N{}_a = \rho_{a_1\dots a_N}$. In linear QM a K-particle subsystem ($K \le N$) is described by observables of the form

$$F^K = g^{N\,ab}F_{a_1\dots a_K}g_{a_{K+1}}\dots g_{a_N}\rho_{b_1\dots b_N} = g^{K\,ab}F_{a_1\dots a_K}\rho_{b_1\dots b_K}, \qquad (16)$$

where

$$\begin{aligned} \rho_{b_1\dots b_K} &= g^{a_{K+1}b_{K+1}}\dots g^{a_Kb_N}g_{a_{K+1}}\dots g_{a_N}\rho_{b_1\dots b_Kb_{K+1}\dots b_N}\\ &= g^{b_{K+1}\dots b_N}\rho_{b_1\dots b_Kb_{K+1}\dots b_N} && (17)\end{aligned}$$

is the subsystem's reduced density matrix. Consider now two, M- and $(N-M-K)$-particle, subsystems which do not overlap (i.e. no particle belongs to both of them). Their reduced density matrices are

$$\begin{aligned} \rho^I{}_d &= \rho^I{}_{d_1\dots d_M} = \rho_{d_1\dots d_Md_{M+1}\dots d_N}g^{d_{M+1}\dots d_N}, && (18)\\ \rho^{II}{}_e &= \rho^{II}{}_{e_{M+K+1}\dots e_N} = g^{e_1\dots e_{M+K}}\rho_{e_1\dots e_{M+K}e_{M+K+1}\dots e_N} && (19)\end{aligned}$$

then

Theorem 3

$$\{\rho^I{}_d, \rho^{II}{}_e, \cdot\}^N = 0. \tag{20}$$

The N-particle triple bracket is defined in terms of the N-particle structure constants which are of the one-particle form but now with all g's replaced by g^N's [2]. Theorem 3 implies

Theorem 4 *Consider two, in general nonlinear, observables $F^I[\rho] = F^I[\rho^I]$, $G^{II}[\rho] = G^{II}[\rho^{II}]$ corresponding to two nonoverlapping, M- and $(N-M-K)$-particle subsystems of a larger N-particle system. Then $\{F^I, G^{II}, \cdot\}^N = 0$.*

The meaning of Theorem 4 is the following. Let a composite system consisting of two noninteracting subsystems be described by a (possibly nonlinear) Hamiltonian function $H[\rho] = H^I[\rho^I] + H^{II}[\rho^{II}]$. Then, for *any* S $i\partial_s F^I = \{F^I, H, S\} = \{F^I, H^I, S\}$ and the dynamics of a subsystem is generated by the Hamiltonian function of this subsystem. Theorem 4 is a general result stating that the triple-bracket scheme allows for a consistent composition of N-particle dynamics from elementary single-particle ones. It follows that the density matrix formalism, as opposed to the standard nonlinear Schrödinger equation pure-state framework, does not introduce any new "threshold phenomena" in transition from N to $N+1$ particle systems (cf. [4]).

3 Convexity principle and nonlinearity: A few remarks

A density matrix is usually thought of as a kind of mixture of fundamental (quantum) and ordinary (classical) probabilities. As such it is typically attributed to ensembles of many particles as opposed to a state vector which, at least in some interpretations, may be regarded as a property of a single system. This perspective suggests that a role of density matrices should be reduced to this of a simple mathematical tool allowing for mixing a classical lack of knowledge with fundamental quantum probabilities. Mathematically this seems to imply that the Liouville-von Neumann equation must be linear even if pure states evolve nonlinearly. This point of view forms an implicit philosophical basis of Mielnik's formalism [5] which on one hand does not exclude nonlinear evolutions of pure states forming the boundary of a "figure of states", and on the other requires the figure to be convex.

The triple bracket formalism leads to a weaker form of the convexity principle [6] which can be formulated as the following

Theorem 5 *Let $\rho_0 = \sum_{k=1}^{\infty} p_k(0)|k,0\rangle\langle k,0|$ be a density matrix acting in a separable Hilbert space, and let $\rho(t)$ be a Hermitian solution of a triple bracket equation with H and $S = S(C_1, \dots, C_k, \dots)$ arbitrary. If $\rho(0) = \rho_0$ then for any t there exists a basis $|k, t, \{p_k(0)\}\rangle$ such that $\rho(t) = \sum_{k=1}^{\infty} p_k(0)|k, t, \{p_k(0)\}\rangle\langle k, t, \{p_k(0)\}|$.*

The nonlinearity is manifested in the dependence of $|k, t, \{p_k(0)\}\rangle$ on , $\{p_k(0)\}$. The figure of states is now no longer convex but the eigenvalues of the density matrix can be nevertheless interpreted in the standard way. The problem arises whether one can obtain such a dynamics in a typical counting experiment where the ensemble in question consists of separately arriving particles. Our guess is that this should not be the case and that the nonlinear evolution has to correspond to more complicated physical situations. In a more general perspective we are inclined to depart from the usual interpretation and regard density matrices as more fundamental than state vectors. Some particular cases might then correspond to classical mixtures in analogy to the role played in quantum mechanics by Abelian subalgebras of observables.

4 Appendix

The bispinor convention we use is the following. To any Greek index there corresponds a pair of Latin ones written down in a lexicographic order. For example

$$F_\alpha{}^{\beta'}{}_\gamma = \begin{pmatrix} F_A{}^{B'}{}_C \\ F_A{}^{B'}{}_{C'} \\ F_A{}^B{}_C \\ F_A{}^B{}_{C'} \\ F_{A'}{}^{B'}{}_C \\ \vdots \end{pmatrix}; \quad \varepsilon_\alpha{}^{\beta'} = \begin{pmatrix} \varepsilon_A{}^{B'} \\ \varepsilon_A{}^B \\ \varepsilon_{A'}{}^{B'} \\ \varepsilon_{A'}{}^B \end{pmatrix} = \begin{pmatrix} 0 \\ \varepsilon_A{}^B \\ \varepsilon_{A'}{}^{B'} \\ 0 \end{pmatrix}.$$

Any permutation preserving the lexicographic rule induces a natural isomorphism, say, $F_{\alpha}{}^{\beta'}{}_{\gamma} \to F_{\alpha'}{}^{\beta'}{}_{\gamma}$ where the latter bispinor would begin with $F_{A'}{}^{B'}{}_{C}$. In particular

$$g_{\alpha}{}^{\beta'} = \begin{pmatrix} 0 \\ -\varepsilon_{A}{}^{B} \\ \varepsilon_{A'}{}^{B'} \\ 0 \end{pmatrix}; \quad g_{\alpha'}{}^{\beta} = \begin{pmatrix} 0 \\ \varepsilon_{A'}{}^{B'} \\ -\varepsilon_{A}{}^{B} \\ 0 \end{pmatrix}; \quad g_{\alpha}{}^{\beta} = \begin{pmatrix} -\varepsilon_{A}{}^{B} \\ 0 \\ 0 \\ \varepsilon_{A'}{}^{B'} \end{pmatrix}; \quad g_{\alpha'}{}^{\beta'} = \begin{pmatrix} \varepsilon_{A'}{}^{B'} \\ 0 \\ 0 \\ -\varepsilon_{A}{}^{B} \end{pmatrix}.$$

The bispinor summation convention is illustrated by $G^{\alpha}H_{\alpha} = G^{A}H_{A} + G^{A'}H_{A'} = G^{\alpha'}H_{\alpha'}$.

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